

Stimulated Neutrino Transformation in Supernovae

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Large amplitude oscillations between states of a quantum system can be stimulated by sinusoidal external potentials with frequencies that are equal to the energy level splitting of the states or a fraction thereof. We investigate this effect for neutrino oscillations both analytically and numerically finding a simple expression for amplitude and wavelength of the transitions as a function of the density, the amplitude and wavenumber of the fluctuation, and the matter mixing matrix elements. We apply our findings to the supernova environment and find that it is possible to predict stimulated transitions that occur due to sinusoidal perturbations in the density.

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INTRODUCTION

Neutrino flavor transformation is a complex phenomenon: the general coherent neutrino state is a linear combination of several states with different energy and the oscillations between these states are affected by the presence of matter and may exhibit many-body effects, e.g. [1], if the neutrino density is sufficiently high. Yet despite this complexity, neutrino flavor transformation is a quantum phenomenon so that one should also expect an additional type of transformation: stimulated transitions driven by perturbing potentials with appropriate frequencies. This possibility of driving transitions between the neutrino states was recognized many years ago, e.g. [2–4], when periodic potentials were studied for their relevance for neutrinos passing through the Earth and the Sun. It was found, e.g. [5], that large amplitude transitions - called parametric resonances - occur between the neutrino states when the frequency of the perturbation, k_* , matches an integer fraction of the frequency associated with one of the energy splittings, i.e. $n k_* = \delta k_{ab}$. This result is intriguing because any density profile can be decomposed into a Fourier series, even non-periodic potentials e.g. turbulence [6, 7], and some terms in the series may have wavelengths which are capable of driving parametric resonance transitions. But even if such a mode is possessed by the profile, in order to observe the transition the neutrino must traverse a distance $2\pi/\kappa_n$, where $1/\kappa_n$ is the reduced wavelength of the stimulated transition at the parametric resonance. This wavelength is a function of both the amplitude of the density perturbation, i.e. the Fourier coefficient, the perturbation wavelength $1/k_*$ and the integer n . But one needs relatively high densities and/or large amplitude fluctuations for an appreciable effect to occur over a reasonable distance and such conditions are not found in the Earth or in the Sun.

The combination of large amplitude ($\sim 10\%$) density fluctuations and high densities can be found in supernovae and supernova density fluctuations were found numerically to have a large impact on neutrino flavor evolution [8–

10]. With this motivation, in this letter we revisit the phenomenon of stimulated transition for the supernova environment. We first derive expressions for the reduced wavelength and the amplitude of stimulated transitions both on and off the parametric resonance. Next, to illustrate our findings, we examine the case of a specific supernova density profile which includes a forward shock to which we add a perturbing sinusoidal fluctuation. Using this scenario, we predict the places where the transitions will occur and then verify our predictions with a numerical three neutrino oscillation calculation. Our example demonstrates that it is possible to identify fluctuation wavelengths that will induce large amplitude transitions in the supernova environment.

STIMULATED NEUTRINO TRANSFORMATION

We are interested in the probability that some initial neutrino state $|\nu(x)\rangle$ at x is later detected as the state $|\nu(x')\rangle$ at x' . These probabilities are computed from the S -matrix which relates the initial and final states via $|\nu(x')\rangle = S(x', x)|\nu(x)\rangle$. The probabilities we calculate depend upon the basis and to avoid the intrinsic oscillatory behavior in the flavor basis we prefer to use the instantaneous eigenstates of the Hamiltonian H , known as the matter basis. The two bases are related by the unitary transformation matrix U . For three flavors/mass states the matrix U is parametrized by three mixing angles, θ_{12} , θ_{13} and θ_{23} , a CP phase and two Majoranna phases. The differential equation for the neutrino S -matrix [11, 12] is simply

$$i \frac{dS}{dx} = H S \quad (1)$$

and in the flavor basis the Hamiltonian is

$$H^{(f)} = U_0 K_0^{(m)} U_0^\dagger + V^{(f)} \quad (2)$$

where $K_0^{(m)}$ is a diagonal matrix in the vacuum, U_0 the vacuum mixing matrix, and $V^{(f)}(x)$ some ‘potential’ that we allow to be position dependent. Our goal is to solve Eq. (1) for S given some potential $V(x)$. The problem we

have in mind is the case where $V(x)$ possesses some sort of ‘smooth’ component we denote by \check{V} and a perturbation δV . Thus the flavor basis Hamiltonian is split into three terms, $H^{(f)} = U_0 K_0^{(m)} U_0^\dagger + \check{V}^{(f)} + \delta V^{(f)}$, which we group as $H^{(f)} = \check{H}^{(f)} + \delta V^{(f)}$. The matter states are defined as those which diagonalize $H^{(f)}$ i.e. if $K^{(m)}$ is the diagonal matrix of eigenvalues of H then U is $H^{(f)} = U K^{(m)} U^\dagger$. Similarly, the unperturbed matter states are defined as those which diagonalize $\check{H}^{(f)}$ i.e. $\check{H}^{(f)} = \check{U} \check{K}^{(m)} \check{U}^\dagger$ where $\check{K}^{(m)}$ is the diagonal matrix of unperturbed eigenvalues, $\check{k}_1, \check{k}_2, \dots$, of \check{H} . In this basis

$$H^{(\check{m})} = \check{K}^{(\check{m})} - i\check{U}^\dagger \frac{d\check{U}}{dx} + \check{U}^\dagger \delta V^{(f)} \check{U} \quad (3)$$

We now write the S -matrix for the matter basis as the product $S^{(\check{m})} = \check{S} A$ where \check{S} is defined to be the solution of

$$i \frac{d\check{S}}{dx} = \left[\check{K}^{(\check{m})} - i\check{U}^\dagger \frac{d\check{U}}{dx} \right] \check{S}. \quad (4)$$

If we know the solution to the unperturbed problem, \check{S} , we can solve for the effect of the perturbation by finding the solution to the differential equation for A :

$$i \frac{dA}{dx} = \check{S}^\dagger \check{U}^\dagger \delta V^{(f)} \check{U} \check{S} A. \quad (5)$$

In general the term $\check{U}^\dagger \delta V^{(f)} \check{U}$ which appears in this equation possesses both diagonal and off-diagonal elements. The diagonal elements are easily removed by writing the matrix A as $A = W B$ where $W = \exp(-i\Xi)$ and Ξ a diagonal matrix $\Xi = \text{diag}(\xi_1, \xi_2, \dots)$. Substitution into equation (5) gives a differential equation for B

$$i \frac{dB}{dx} = W^\dagger \left[\check{S}^\dagger \check{U}^\dagger \delta V^{(f)} \check{U} \check{S} - \frac{d\Xi}{dx} \right] W B \quad (6)$$

and Ξ is chosen so that $d\Xi/dx$ removes the diagonal elements of $\check{S}^\dagger \check{U}^\dagger \delta V^{(f)} \check{U} \check{S}$. Once Ξ has been found, determining transition probabilities involves solving dB/dx .

CONSTANT POTENTIALS WITH SINUSOIDAL PERTURBATIONS

We now consider the specific case of a matter potential, so that the only non-zero entry of $\check{V}^{(f)}$ is \check{V}_{ee} and $\check{V}_{ee}(x)$ is $\check{V}_{ee}(x) = \sqrt{2} G_F \check{n}_e(x)$ where \check{n}_e is the ‘smooth’ electron density. With this form for the potential the elements of $\check{U}^\dagger \check{V}^{(f)} \check{U} = \check{V}^{(\check{m})}$ are $\check{V}_{ij}^{(\check{m})} = \check{U}_{ei}^\dagger \check{U}_{ej} V_{ee}$. Next we Taylor expand $\check{V}_{ee}(x)$ as $\check{V}_{ee}(x) = V_\star + \dots$ and retain only the first term of the expansion. For constant $\check{V}_{ee}(x) = V_\star$ the S -matrix describing the unperturbed problem, \check{S} , is diagonal

because the eigenvalues of \check{H} and thus the mixing matrix \check{U} are fixed so \check{S} is simply $\check{S} = \exp(-i\check{K}^{(\check{m})} x)$. The perturbation we consider is the case of a single sinusoidal fluctuation of wavenumber k_\star , amplitude C_\star and phase shift $k_\star x_\star$ i.e. $\delta V_{ee}(x) = C_\star V_\star \sin[k_\star(x + x_\star)]$. With this perturbing potential we solve for the quantities ξ_i :

$$\xi_i = \frac{C_\star V_\star |\check{U}_{ei}|^2}{k_\star} [\cos(k_\star x_\star) - \cos(k_\star [x + x_\star])]. \quad (7)$$

Everything we require to start solving the equation for dB/dx has now been defined. For just two flavors the differential equation for the matrix B is

$$i \frac{dB}{dx} = C_\star V_\star \sin(k_\star [x + x_\star]) \begin{pmatrix} 0 & \check{U}_{e1}^\dagger \check{U}_{e2} e^{i[\delta \check{k}_{12} x + \delta \xi_{12}]} \\ \check{U}_{e2}^\dagger \check{U}_{e1} e^{-i[\delta \check{k}_{12} x + \delta \xi_{12}]} & 0 \end{pmatrix} B \quad (8)$$

where $\delta \check{k}_{12} = \check{k}_1 - \check{k}_2$ and similarly $\delta \xi_{12} = \xi_1 - \xi_2$. Our first step in solving this equation is to introduce the quantities z_{12} and κ_n , defined to be

$$z_{12} = \frac{C_\star V_\star}{k_\star} (|\check{U}_{e1}|^2 - |\check{U}_{e2}|^2), \quad (9)$$

$$\kappa_n = (-i)^{n-1} \frac{n C_\star V_\star}{z_{12}} J_n(z_{12}) \check{U}_{e1}^\dagger \check{U}_{e2} \times e^{i[n k_\star x_\star + z_{12} \cos(k_\star x_\star)]} \quad (10)$$

respectively where J_n is the Bessel J function, and our second is to use the Jacobi-Anger expansion for $\exp(i\delta \xi_{12})$

$$\exp(i\delta \xi_{12}) = \exp[i z_{12} \cos(k_\star x_\star)] \sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{12}) \exp[i n k_\star (x + x_\star)] \quad (11)$$

With this expansion and two definitions we find equation (8) becomes

$$i \frac{dB}{dx} = i \sum_n \begin{pmatrix} 0 & -\kappa_n e^{i[\delta \check{k}_{12} + n k_\star] x} \\ \kappa_n^\star e^{-i[\delta \check{k}_{12} + n k_\star] x} & 0 \end{pmatrix} B \quad (12)$$

which is a greatly simplified equation. We now adopt the Rotating Wave Approximation and drop all terms in the sum except the one closest to parametric resonance i.e. where $\delta \check{k}_{12} + n k_\star \approx 0$. With this approximation we find that equation (12) falls into a class of matrix Schrodinger equations with known solutions. Introducing $2k_n = \delta \check{k}_{12} + n k_\star$ and $q_n^2 = k_n^2 + \kappa_n^2$, our final result is that

$$B = \begin{pmatrix} e^{i k_n x} \left[\cos(q_n x) - i \frac{k_n}{q_n} \sin(q_n x) \right] & -e^{i k_n x} \frac{\kappa_n}{q_n} \sin(q_n x) \\ e^{-i k_n x} \frac{\kappa_n^\star}{q_n} \sin(q_n x) & e^{-i k_n x} \left[\cos(q_n x) + i \frac{k_n}{q_n} \sin(q_n x) \right] \end{pmatrix}. \quad (13)$$

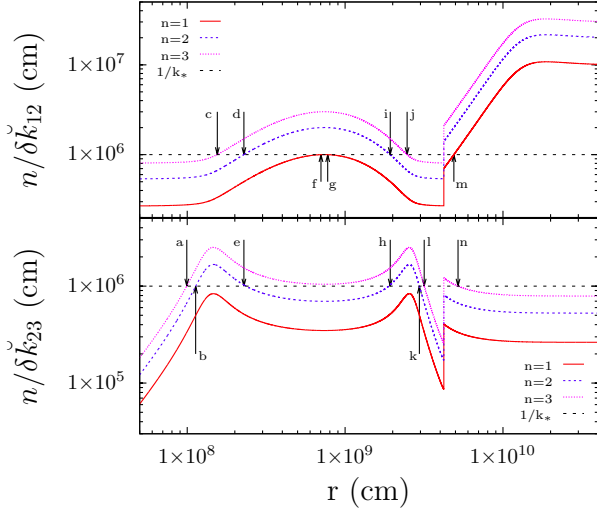


FIG. 1: Using the base density profile described in the text, we plot multiples of the wavelength associated with the mass splittings, $n/\delta k_{12}$ (upper panel) and $n/\delta k_{23}$ (lower panel) as a function of distance. In each panel, the lowest solid curve represents $n=1$, the middle is $n=2$ and highest is $n=3$. The wavelength of a sinusoidal perturbing potential with reduced wavelength $\lambda_* = 1/k_* = 10$ km is plotted as the dashed line. When any of the solid lines crosses the dashed line, the parametric resonance condition is fulfilled and thus there exists possibility of a stimulated transition.

Because both \check{S} and W are diagonal matrices, from this equation we can read off the transition probability between the matter states 1 and 2

$$P_{12} = \frac{\kappa_n^2}{q_n^2} \sin^2(q_n x). \quad (14)$$

The range of k_* such that the amplitude of the oscillation is greater than $1/2$ is $\Delta k_* = 4|\kappa_n|/n$ and we see that this width is proportional to C_*^n when the parameter z_{12} is small. In summary then, we expect to see in $P_{12}(k_*)$ a series of peaks at the undertones of $|\delta k_{12}|$ with widths that decrease geometrically. This same behavior has been found for irradiated two-level molecules with permanent dipoles [13].

APPLICATION TO THE SUPERNOVA ENVIRONMENT

We demonstrate the phenomenon of stimulated transition in supernova-like conditions by constructing a density vs. radius relationship from the parametrized form from Fogli *et al.* [10]. The shock position was set at $t = 4$ s and we superpose on the profile a single sinusoid perturbation with a reduced wavelength $\lambda_* = 1/k_* = 10$ km and amplitude $C_* = 0.1$. Through this profile we send a neutrino with the following properties, $\delta m_{12}^2 = 3 \times 10^{-3} \text{eV}^2$, $\delta m_{23}^2 = 8 \times 10^{-5} \text{eV}^2$, $\theta_{12} = 33^\circ$, $\theta_{13} = 9^\circ$, $\theta_{23} = 45^\circ$, $\delta = 0$ and with an energy of 20 MeV.

Using the base density profile and the parametric resonance condition, $\delta k_{ij} + n k_* = 0$, where the subscripts i and j denotes matter states 1,2 or 3, we predict the locations where significant stimulated transitions between the states may occur. These are the places where the $n/\delta k_{ij}$ curves in Fig. (1), intersect the horizontal dashed line corresponding to λ_* . We see that there are seven locations (c,d,f,g,i,j,m) in the 1,2 channel (top panel) and seven locations (a,b,e,h,k,l,n) in the 2,3 channel (bottom panel).

The formalism developed in the previous section assumed a constant density profile, but the supernova-like density profile we have chosen is not constant. Thus we only expect to see stimulated transitions at some subset of the fourteen locations identified in Fig. 1. To determine this subset, can use a comparison of the density scale height $r_\rho = |\rho/(d\rho/dr)|$ with the reduced parametric resonance wavelength $1/q_n$. When $r_\rho \gg 1/q_n$, then the constant density approximation is reasonable, and we should expect to see stimulated transitions. In Fig. 2 where the top (bottom) panel again corresponds to the 1,2 (2,3) channel, it can be seen that the stimulated transition wavelength becomes long at the parametric resonance locations. In the 1,2 channel, only for points f,g is $r_\rho \gg 1/q_n$, although r_ρ is slightly larger than $1/q_n$ at m. Thus, we expect a significant stimulated transition at f,g, and a more moderate one at m. In the 2,3 channel, this figure suggests that the most significant transition will be at location k with more moderate transitions at h and l. In Fig. 3 we check these results by evolving the neutrinos through the base + perturbing profile with an exact numerical three neutrino calculation. The base density profile possesses three Mikheev-Smirnov-Wolfenstein (MSW) [14, 15] high (H) density resonances. Two of these, labelled X and Y on the figure, are semiadiabatic while the third, Z, is at the position of the shock and is diabatic. The evolution of the neutrino through the three MSW resonances is accounted for with the unperturbed matrix \check{S} . Note that we have plotted transition probabilities between matter eigenstates not mass eigenstates, so the relevant MSW H transition is in the 2,3 channel. At the expected stimulated transition locations, we indeed see strong transitions at f,g between matter eigenstates 1 and 2, and a somewhat smaller transition in the same channel at m. In the 2,3 channel we see a strong transition at k and smaller transitions at h and l. We see also a few additional features at the parametric resonance locations caused by simultaneous occurrence of stimulated transitions between multiple pairs of states. For example, around points f,g and around m both $1/\delta k_{12} \approx \lambda_*$ and $3/\delta k_{23} \approx \lambda_*$. This dual satisfaction of the parametric resonance conditions implies $4/\delta k_{13} \approx \lambda_*$ which means we can have stimulated transitions between all three states simultaneously.

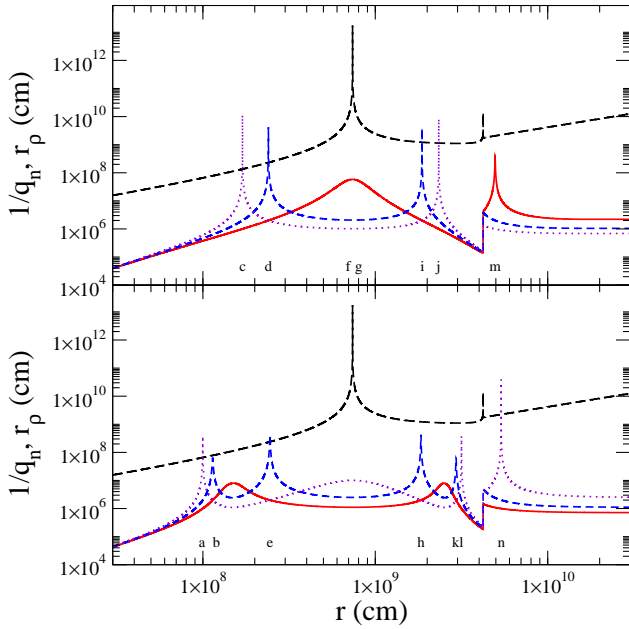


FIG. 2: Comparison of the reduced wavelength $1/q_n$ for the $n=1$, $n=2$ and $n=3$ modes (red, blue and purple lines) with the density scale height, r_p for the base density profile (black dashed line). At the parametric resonances identified in Fig. 1, the transition wavelength $1/q_n$ becomes long. When r_p is larger than $1/q_n$, we expect stimulated transitions.

CONCLUSIONS

We have investigated the effect of sinusoidal density fluctuations upon neutrino propagation and derived an analytic solution for the matter basis transition probabilities in the case of constant density. Large transitions between the states can occur and we have determined their wavelengths and amplitudes. We presented an example calculation with a supernova-like base density profile to which we added a sinusoidal perturbation. When compared with a numerical three neutrino flavor calculation, we found we were able to successfully predict the locations of the stimulated transitions, despite the presence of multiple MSW resonances in the unperturbed profile. This method holds promise as a tool for analyzing consequences of density perturbations on supernova neutrino flavor transformation.

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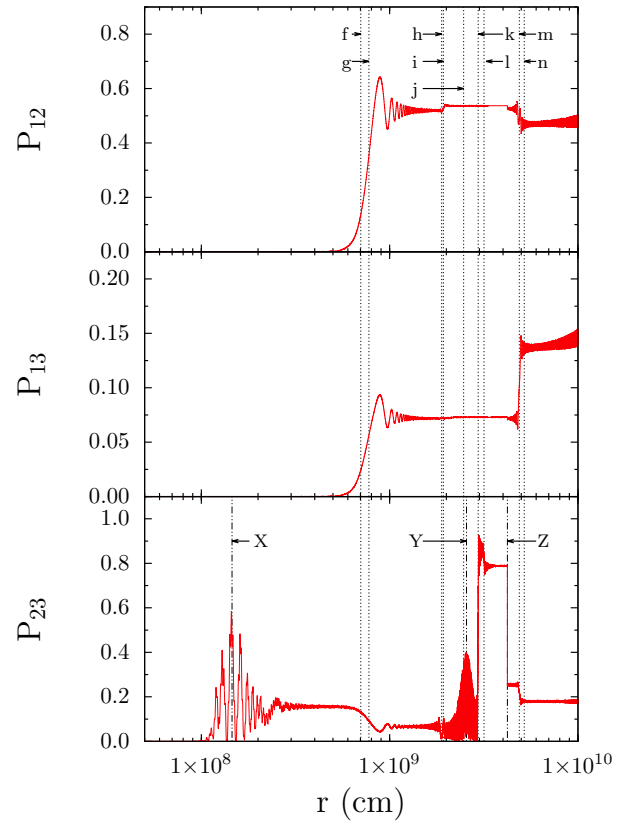


FIG. 3: Transition probabilities between matter eigenstates, P_{12} , P_{13} and P_{23} , as calculated numerically. Large stimulated transitions occur at f and g (top panel) and at k (bottom panel). Smaller amplitude stimulated transitions occur at m, h, and l. The transitions labeled X, Y and Z are base profile MSW H resonances.

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